Chapter 7
Input–Output Formulation of Optical Cavities

Abstract In preceding chapters we have used a master-equation treatment to calculate the photon statistics inside an optical cavity when the internal field is damped. This approach is based on treating the field external to the cavity, to which the system is coupled, as a heat bath. The heat bath is simply a passive system with which the system gradually comes into equilibrium. In this chapter we will explicitly treat the heat bath as the external cavity field, our object being to determine the effect of the intracavity dynamics on the quantum statistics of the output field. Within this perspective we will also treat the field input to the cavity explicitly. This approach is necessary in the case of squeezed state generation due to interference effects at the interface between the intracavity field and the output field.

An input–output formulation is also required if the input field state is specified as other than simply a vacuum or thermal state. In particular, we will want to discuss the case of an input squeezed state.

7.1 Cavity Modes

We will consider a single cavity mode interacting with an external multi-mode field. To being with we will assume the cavity has only one partially transmitting mirror that couples the intracavity mode to the external field. The geometry of the cavity and the nature of the dielectric interface at the mirror determines which output modes couple to the intracavity mode. It is usually the case that the emission is strongly direction. We will assume that the only modes that are excited have the same plane polarisation and are all propagating in the same direction, which we take to be the positive x-direction. The positive frequency components of the quantum electric field for these modes are then

$$E^{(+)}(x,t) = \mathrm{i}\sum_{n=0}^{\infty}\left(\frac{\hbar\omega_n}{2\varepsilon_0 V}\right)^{1/2} b_n \mathrm{e}^{-\mathrm{i}\omega_n(t-x/c)} \tag{7.1}$$

In ignoring all the other modes, we are implicitly assuming that they remain in the vacuum state.

Let us further assume that all excited modes of this form have frequencies centered on the cavity resonance frequency and we call this the *carrier frequency* of $\Omega >> 1$. Then we can approximate the positive frequency components by

$$E^{(+)}(x,t) = \mathrm{i}\left(\frac{\hbar\Omega_n}{2\varepsilon_0 Ac}\right)^{1/2}\sqrt{\frac{c}{L}}\sum_{n=0}^{\infty} b_n \mathrm{e}^{-\mathrm{i}\omega_n(t-x/c)} \tag{7.2}$$

where A is a characteristic transverse area. This operator has dimensions of electric field. In order to simplify the dimensions we now define a field operator that has dimensions of $\mathrm{s}^{-1/2}$. Taking the continuum limit we thus define the positive frequency operator for modes propagating in the *positive x*–direction,

$$b(x,t) = \mathrm{e}^{-\mathrm{i}\Omega(t-x/c)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega b(\omega)\mathrm{e}^{-\mathrm{i}\omega(t-x/c)} \tag{7.3}$$

where we have made a change of variable $\omega \mapsto \Omega + \omega'$ and used the fact that $\Omega >> 1$ to set the lower limit of integration to minus infinity, and

$$[b(\omega_1), b^\dagger(\omega_2)] = \delta(\omega_1 - \omega_2) \tag{7.4}$$

In this form the moment $n(x,t) = \langle b^\dagger(x,t)b(x,t)\rangle$ has units of s^{-1}. This moment determines the probability per unit time (the count rate) to count a photon at space-time point (x,t).

Consider now the single side cavity geometry depicted in Fig. 7.1. The field operators at some external position, $b(t) = b(x > 0,t)\mathrm{e}^{\mathrm{i}\Omega t}$ and $b^\dagger(t) = b^\dagger(x > 0,t)\mathrm{e}^{-\mathrm{i}\Omega t}$ can be taken to describe the field, in the interaction picture with frequency Ω. As the cavity is confined to some region of space, we need to determine how the field outside the cavity responds to the presence of the cavity and any matter it may contain. The interaction Hamiltonian between the cavity field, represented by the harmonic oscillator annihilation and creation operators a, $a^\dagger$, and the external field in the rotating wave approximation is given by (6.14). Restricting the sum to only the modes of interest and taking the continuum limit, we can write this as

$$V(t) = i\hbar\int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)[b(\omega)a^\dagger - ab^\dagger(\omega)] \tag{7.5}$$

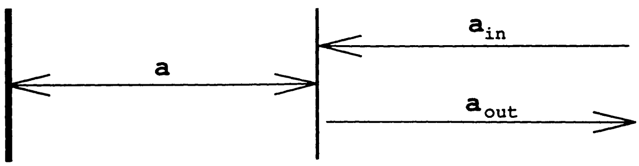

Fig. 7.1 A schematic representation of the cavity field and the input and output fields for a single-sided cavity

with $[a, a^\dagger] = 1$ and $g(\omega)$ is the coupling strength as a function of frequency which is typically peaked around $\omega = 0$ (which corresponds to $\omega = \Omega$ in the original non-rotating frame). In fact $g(\omega)$ is the Fourier transform of a spatially varying coupling constant that describes the local nature of the cavity/field interaction (see [1]). If the cavity contains matter, the field inside the cavity may acquire some non trivial dynamics which then forces the external fields to have a time dependence different from the free field dynamics. This leads to an explicit time dependence in the frequency space operators, $b(t,\omega)$, in the Heisenberg picture.

We now follow the approach of *Collett and Gardiner* [1]. The Heisenberg equation of motion for $b(t,\omega)$, in the interaction picture, is

$$\dot{b}(t,\omega) = -\mathrm{i}\omega b(\omega) + g(\omega)a \tag{7.6}$$

The solution to this equation can be written in two ways depending on weather we choose to solve in terms of the initial conditions at time $t_0 < t$ (the *input*) or in terms of the final conditions at times $t_1 > t$, (the *output*). The two solutions are respectively

$$b(t,\omega) = \mathrm{e}^{-i\omega(t-t_0)} b_0(\omega) + g(\omega) \int_0^t \mathrm{e}^{-\mathrm{i}\omega(t-t')} a(t')\mathrm{d}t' \tag{7.7}$$

where $t_0 < t$ and $b_0(\omega) = b(t = t_0, \omega)$, and

$$b(t,\omega) = \mathrm{e}^{-\mathrm{i}\omega(t-t_1)} b_1(\omega) - g(\omega) \int_t^{t_1} \mathrm{e}^{-\mathrm{i}\omega(t-t')} a(t')\mathrm{d}t' \tag{7.8}$$

where $t < t_1$ and $b_1(\omega) = b(t = t_1, \omega)$. In physical terms $b_0(\omega)$ and $b_1(\omega)$ are usually specified at $-\infty$ and $+\infty$ respectively, that is, for times such that the field is simply a free field, however here we only require $t_0 < t < t_1$.

The cavity field operator obeys the equation

$$\dot{a} = -\frac{\mathrm{i}}{h}[\mathscr{H}_S, a] - \int_{-\infty}^{\infty} \mathrm{d}\omega\, g(\omega) b(t,\omega) \tag{7.9}$$

where $\mathscr{H}_{\mathscr{S}}$ is the Hamiltonian for the cavity field alone. In terms of the solution with initial conditions, (7.7), this equation becomes

$$\begin{aligned} \dot{a} = -\frac{\mathrm{i}}{\hbar}[\mathscr{H}_S, a] &- \int_{-\infty}^{\infty} \mathrm{d}\omega\, g(\omega) \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} b_0(\omega) \\ &- \int_{-\infty}^{\infty} \mathrm{d}\omega\, g(\omega)^2 \int_{t_0}^{t} \mathrm{e}^{-\mathrm{i}\omega(t-t')} a(t') \end{aligned} \tag{7.10}$$

We now assume that $g(\omega)$ is independent of frequency over a wide range of frequencies around $\omega = 0$ (that is around $\omega = \Omega$ in non rotating frame). This is the first approximation we need to get a Markov quantum stochastic process. Thus we set

$$g(\omega)^2 = \gamma/2\pi \tag{7.11}$$

We also define an *input field* operator by

$$a_{\mathrm{IN}}(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} b_0(\omega) \tag{7.12}$$

(the minus sign is a phase convention: left-going fields are negative, right-going fields are positive). Using the relation

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \mathrm{e}^{-\mathrm{i}\omega(t-t')} = 2\pi\delta(t-t') \tag{7.13}$$

the input field may be shown to satisfy the commutation relations

$$[a_{\mathrm{IN}}(t), a_{\mathrm{IN}}^{\dagger}(t') = \delta(t-t') \tag{7.14}$$

When (7.13) is achieved as the limit of an integral of a function which goes smoothly to zero at $\pm\infty$ (for example, a Gaussian), the following result also holds

$$\int_{t_0}^{t} f(t')\delta(t-t')\mathrm{d}t' = \int_{t}^{t_1} f(t')\delta(t-t')\mathrm{d}t' = \frac{1}{2}f(t), \quad (t_0 < t < t_1) \tag{7.15}$$

Interchanging the order of time and frequency integration in the last term in (7.10) and using (7.15) gives

$$\dot{a}(t) = -\frac{\mathrm{i}}{\hbar}[a(t), \mathscr{H}_{\mathrm{SYS}}] - \frac{\gamma}{2}a(t) + \sqrt{\gamma}a_{\mathrm{IN}}(t) \tag{7.16}$$

Equation (7.16) is a *quantum stochastic differential equation* (qsde) for the intracavity field, $a(t)$. The quantum noise term appears explicitly as the input field to the cavity.

In a similar manner we may substitute the solution in terms of final conditions, (7.8) into (7.10) to obtain the time-reversed qsde as

$$\dot{a}(t) = -\frac{\mathrm{i}}{\hbar}[a(t), \mathscr{H}_{\mathrm{SYS}}] + \frac{\gamma}{2}a(t) - \sqrt{\gamma}a_{\mathrm{IN}}(t) \tag{7.17}$$

where we define the output field operator as

$$a_{\mathrm{OUT}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega \mathrm{e}^{-\mathrm{i}\omega(t-t_1)} b_1(\omega) \tag{7.18}$$

(Note that the phase convention between left going and right going external fields required for the boundary condition has been explicitly incorporated in the definitions of $a_{\mathrm{IN}},\ a_{\mathrm{OUT}}$). The input and output fields are then seen to be related by

$$a_{\mathrm{IN}}(t) + a_{\mathrm{OUT}}(t) = \sqrt{\gamma} a(t) \tag{7.19}$$

This represents a boundary condition relating each of the far field amplitudes outside the cavity to the internal cavity field. Interference terms between the input and the cavity field may contribute to the observed moments when measurements are made on a_{OUT}.

7.2 Linear Systems

For many systems of interest the Heisenberg equations of motion are linear and may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}(t) = \boldsymbol{A}a(t) - \frac{\gamma}{2}\boldsymbol{a}(t) + \sqrt{\gamma}\boldsymbol{a}_{\mathrm{IN}}(t)\,, \tag{7.20}$$

where

$$\boldsymbol{a}(t) = \begin{pmatrix} a(t) \\ a^{\dagger}(t) \end{pmatrix}\,, \tag{7.21}$$

$$\boldsymbol{a}_{\mathrm{IN}}(t) = \begin{pmatrix} a_{\mathrm{IN}}(t) \\ a_{\mathrm{IN}}^{\dagger}(t) \end{pmatrix}\,, \tag{7.22}$$

Define the Fourier components of the intracavity field by

$$a(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(t-t_0)} a(\omega) \mathrm{d}\omega \tag{7.23}$$

and a frequency component vector

$$\boldsymbol{a}(\omega) = \begin{pmatrix} a(\omega) \\ a^{\dagger}(\omega) \end{pmatrix} \tag{7.24}$$

where $a^{\dagger}(\omega)$ is the Fourier transform of $a^{\dagger}(t)$.

The equations of motion become

$$\left[\boldsymbol{A} + \left(\mathrm{i}\omega - \frac{\gamma}{2}\right)\boldsymbol{I}\right]\boldsymbol{a}(\omega) = -\sqrt{\gamma}\boldsymbol{a}_{\mathrm{IN}}(\omega)\,. \tag{7.25}$$

However, we may use (7.18) to eliminate the internal modes to obtain

$$\boldsymbol{a}_{\mathrm{OUT}}(\omega) = -\left[\boldsymbol{A} + \left(\mathrm{i}\omega + \frac{\gamma}{2}\right)\boldsymbol{1}\right]\left[\boldsymbol{A} + \left(\mathrm{i}\omega - \frac{\gamma}{2}\right)\boldsymbol{1}\right]^{-1}\boldsymbol{a}_{\mathrm{IN}}(\omega)\,. \tag{7.26}$$

To illustrate the use of this result we shall apply it to the case of an empty one-sided cavity. In this case the only source of loss in the cavity is through the mirror which couples the input and output fields. The system Hamiltonian is

$$\mathscr{H}_{\mathrm{SYS}} = \hbar\omega_0 a^\dagger a\,.$$

Thus

$$\boldsymbol{A} = \begin{pmatrix} -\mathrm{i}\omega_0 & 0 \\ 0 & \mathrm{i}\omega_0 \end{pmatrix}\,. \tag{7.27}$$

Equation (7.26) then gives

$$\boldsymbol{a}_{\mathrm{OUT}}(\omega) = \frac{\frac{\gamma}{2} + \mathrm{i}(\omega - \omega_0)}{\frac{\gamma}{2} - \mathrm{i}(\omega - \omega_0)}\boldsymbol{a}_{\mathrm{IN}}(\omega)\,. \tag{7.28}$$

Thus there is a frequency dependent phase shift between the output and input. The relationship between the input and the internal field is

$$\boldsymbol{a}(\omega) = \frac{\sqrt{\gamma}}{\frac{\gamma}{2} - \mathrm{i}(\omega - \omega_0)} a_{\mathrm{IN}}(\omega)\,, \tag{7.29}$$

which leads to a Lorentzian of width $\gamma/2$ for the intensity transmission function.

7.3 Two-Sided Cavity

A two-sided cavity has two partially transparent mirrors with associated loss coefficients γ_1 and γ_2, as shown in Fig. 7.2. In this case there are two input ports and two output ports. The equation of motion for the internal field is then given by an obvious generalisation as

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -\mathrm{i}\omega_0 a(t) - \frac{1}{2}(\gamma_1 + \gamma_2)a(t) + \sqrt{\gamma_1}a_{\mathrm{IN}}(t) + \sqrt{\gamma_2}b_{\mathrm{IN}}(t)\,. \tag{7.30}$$

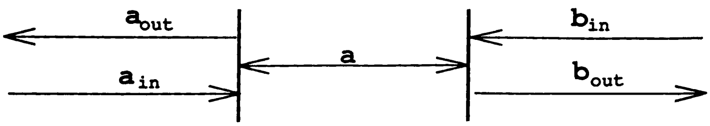

Fig. 7.2 A schematic representation of the cavity field and the input and output fields for a double-sided cavity

The relationship between the internal and input field frequency components for an empty cavity is then

$$\boldsymbol{a}(\omega) = \frac{\sqrt{\gamma_1}\boldsymbol{a}_{\mathrm{IN}}(\omega) + \sqrt{\gamma_2}\boldsymbol{b}_{\mathrm{IN}}(\omega)}{\left(\frac{\gamma_1+\gamma_2}{2}\right) - \mathrm{i}(\omega-\omega_0)} . \tag{7.31}$$

The relationship between the input and output modes may be found using the boundary conditions at each mirror, see (7.19),

$$a_{\mathrm{OUT}}(t) + a_{\mathrm{IN}}(t) = \sqrt{\gamma_1}a(t) , \tag{7.32a}$$

$$b_{\mathrm{OUT}}(t) + b_{\mathrm{IN}}(t) = \sqrt{\gamma_2}a(t) . \tag{7.32b}$$

We find

$$\boldsymbol{a}_{\mathrm{OUT}}(\omega) = \frac{\left[\frac{\gamma_1-\gamma_2}{2} + \mathrm{i}(\omega-\omega_0)\right]\boldsymbol{a}_{\mathrm{IN}}(\omega) + \sqrt{\gamma_1\gamma_2}\boldsymbol{b}_{\mathrm{IN}}(\omega)}{\frac{\gamma_1+\gamma_2}{2} - \mathrm{i}(\omega-\omega_0)} \tag{7.33}$$

For equally reflecting mirrors $\gamma_1 = \gamma_2 = \gamma$ this expression simplifies to

$$\boldsymbol{a}_{\mathrm{OUT}}(\omega) = \frac{\mathrm{i}(\omega-\omega_0)\boldsymbol{a}_{\mathrm{IN}}(\omega) + \gamma\boldsymbol{b}_{\mathrm{IN}}(\omega)}{\gamma - \mathrm{i}(\omega-\omega_0)} . \tag{7.34}$$

Near to resonance this is approximately a through pass Lorentzian filter

$$\boldsymbol{a}_{\mathrm{OUT}}(\omega) \approx \frac{\gamma\boldsymbol{b}_{\mathrm{IN}}(\omega)}{\gamma - \mathrm{i}(\omega-\omega_0)} , \tag{7.35}$$

This is only an approximate result, the neglected terms are needed to preserve the commutation relations. Away from resonance there is an increasing amount of backscatter. In the limit $|\omega-\omega_0| \gg \gamma$ the field is completely reflected

$$\boldsymbol{a}_{\mathrm{OUT}}(\omega) = -\boldsymbol{a}_{\mathrm{IN}}(\omega) . \tag{7.36}$$

Before going on to consider interactions within the cavity we shall derive some general relations connecting the two time correlation functions inside and outside the cavity.

7.4 Two Time Correlation Functions

Integrating (7.7) over ω, and using (7.13) gives

$$\boldsymbol{a}_{\mathrm{IN}}(t) = \frac{\sqrt{\gamma}}{2}a(t) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega b(\omega,t) . \tag{7.37}$$

Let $c(t)$ be any system operator. Then

$$[c(t), \sqrt{\gamma} a_{\mathrm{IN}}(t)] = \frac{\gamma}{2}[c(t), a(t)] \,. \tag{7.38}$$

Now since $c(t)$ can only be a function of $a_{\mathrm{IN}}(t')$ for earlier times $t' < t$ and the input field operators must commute at different times we have

$$[c(t), \sqrt{\gamma} a_{\mathrm{IN}}(t')] = 0, \quad t' > t \,. \tag{7.39}$$

Similarly

$$[c(t), \sqrt{\gamma} a_{\mathrm{OUT}}(t')] = 0, \quad t' < t \,. \tag{7.40}$$

From (7.40 and 7.18) we may show that

$$[c(t), \sqrt{\gamma} a_{\mathrm{IN}}(t')] = \gamma[c(t), a(t)], \quad t' < t \,. \tag{7.41}$$

Combining (7.38–7.41) we then have

$$[c(t), \sqrt{\gamma} a_{\mathrm{IN}}(t')] = \gamma\theta(t - t')[c(t), a(t')] \,, \tag{7.42}$$

where $\theta(t)$ is the step function

$$\theta(t) = \begin{cases} 1 & t > 0, \\ \frac{1}{2} & t = 0, \\ 0 & t < 0. \end{cases} \tag{7.43}$$

The commutator for the output field may now be calculated to be

$$[a_{\mathrm{OUT}}(t), a^{\dagger}_{\mathrm{OUT}}(t')] = [a_{\mathrm{IN}}(t), a^{\dagger}_{\mathrm{IN}}(t')] \tag{7.44}$$

as required.

For the case of a coherent or vacuum input it is now possible to express variances of the output field entirely in terms of those of the internal system. For an input field of this type all moments of the form $\langle a^{\dagger}_{\mathrm{IN}}(t) a_{\mathrm{IN}}(t') \rangle$, $\langle a(t) a_{\mathrm{IN}}(t') \rangle$, $\langle a^{\dagger}(t) a_{\mathrm{IN}}(t') \rangle$, $\langle a^{\dagger}_{\mathrm{IN}}(t) a(t') \rangle$, and $\langle a^{\dagger}_{\mathrm{IN}}(t) a^{\dagger}(t') \rangle$ will factorise. Using (7.18) we find

$$\langle a^{\dagger}_{\mathrm{OUT}}(t), a_{\mathrm{OUT}}(t') \rangle = \gamma \langle a^{\dagger}(t), a(t') \rangle \,, \tag{7.45}$$

where

$$\langle U,\, V \rangle \equiv \langle U\, V \rangle - \langle U \rangle \langle V \rangle \,. \tag{7.46}$$

In this case there is a direct relationship between the two time correlation of the output field and the internal field. Consider now the phase dependent two time correlation function

$$\begin{aligned}\langle a_{\mathrm{OUT}}(t), a_{\mathrm{OUT}}(t')\rangle &= \langle a_{\mathrm{IN}}(t) - \sqrt{\gamma}a(t), a_{\mathrm{IN}}(t') - \sqrt{\gamma}a(t')\rangle \\ &= \gamma\langle a(t), a(t')\rangle - \sqrt{\gamma}\langle [a_{\mathrm{IN}}(t'), a(t)]\rangle \\ &= \gamma\langle a(t), a(t')\rangle + \gamma\theta(t'-t)\langle [a(t'), a(t)]\rangle \\ &= \gamma\langle a(\max(t,t')), a(\min(t,t'))\rangle \ . \end{aligned} \tag{7.47}$$

In this case the two time correlation functions of the output field are related to the time ordered two time correlation functions of the cavity field.

These results mean that the usual spectrum of the output field, as given by the Fourier transform of (7.45), will be identical to the spectrum of the cavity field. The photon statistics of the output field will also be the same as the intracavity field. Where a difference will arise, is in phase-sensitive spectrum such as in squeezing experiments.

7.5 Spectrum of Squeezing

The output field from the cavity is a multi mode field. Phase-dependent properties of this field are measured by mixing the field, on a beam splitter, with a known coherent field – the local oscillator, as discussed in Sect. 3.8. The resulting field may then be directed to a photodetector and the measured photocurrent directed to various devices such as a noise-power spectrum analyser to produce a spectrum, $S(\omega)$. If we write the signal field as $a_{\mathrm{out}}(t)$ and the local oscillator is $a_{\mathrm{LO}}(t)$, the average photo current is proportional to

$$\begin{aligned}\overline{\mathrm{i}(t)} = &(1-\eta)\langle a^{\dagger}_{\mathrm{LO}}(t)a_{\mathrm{LO}}(t)\rangle + \sqrt{\eta(1-\eta)}\langle a_{\mathrm{OUT}}(t)a^{\dagger}_{\mathrm{LO}}(t) + a^{\dagger}_{\mathrm{OUT}}(t)a_{\mathrm{LO}}(t)\rangle \\ &+\eta\langle a^{\dagger}_{\mathrm{OUT}}(t)a_{\mathrm{OUT}}(t)\rangle \end{aligned} \tag{7.48}$$

If $(1-\eta)\langle a^{\dagger}_{\mathrm{LO}}(t)a_{\mathrm{LO}}(t)\rangle >> \eta\langle a^{\dagger}_{\mathrm{OUT}}(t)a_{\mathrm{OUT}}(t)\rangle$, we can neglect the last term in (7.48). If the local oscillator is in a coherent state $\langle a_{\mathrm{LO}}(t)\rangle = |\beta|\mathrm{e}^{\mathrm{i}\theta}\ \mathrm{e}^{-\mathrm{i}\Omega t}$, then not only the average current, but all its moments are determined by the quantum statistics of the quadrature phase operator

$$X^{\mathrm{OUT}}_{\theta} = a_{\mathrm{OUT}}\mathrm{e}^{-\mathrm{i}(\theta-\Omega t)} + a^{\dagger}_{\mathrm{OUT}}\mathrm{e}^{-\mathrm{i}(\theta-\Omega t)} \tag{7.49}$$

In particular, the noise power spectrum of the photocurrent is given by

$$S(\omega,\theta) = \int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{-\mathrm{i}\omega t}\langle :\ X^{\mathrm{OUT}}_{\theta}(t), X^{\mathrm{OUT}}_{\theta}(0)\ :\rangle \tag{7.50}$$

where: indicates normal ordering. The combination $a_{\mathrm{OUT}}\mathrm{e}^{\mathrm{i}\Omega t}$ is simply the definition of the output field in the interaction picture defined at frequency Ω. Using (7.47) and (7.49) this may be written in terms of the intracavity field as

$$S(\omega,\theta) = \gamma \int_{-\infty}^{\infty} \mathrm{d}t \; \mathrm{e}^{-\mathrm{i}\omega t} T \langle : \; X_\theta(t), X_\theta(0) \; : \rangle \tag{7.51}$$

where T denotes time-ordering and $X_\theta(t)$ intracavity quadrature phase operator in an interaction picture at frequency, Ω, defined by the local oscillator frequency,

$$X_\theta(t) = a(t)e^{-i\theta} + a^\dagger(t)e^{i\theta} \tag{7.52}$$

Conventionally we define the in-phase and quadrature-phase operators as $X_1 = X_{\theta=0}$, $X_2 = X_{\theta=\pi/2}$.

7.6 Parametric Oscillator

We shall now proceed to calculate the squeezing spectrum from the output of a parametric oscillator. Below threshold the equations for the parametric oscillator are linear and hence we can directly apply the linear operator techniques. When the equations are nonlinear such as for the parametric oscillator above threshold, then linearization procedures must be used. One procedure using the Fokker–Planck equation is described in Chap. 8.

Below threshold the pump mode of the parametric oscillator may be treated classically. It can then be described by the Hamiltonian

$$\mathscr{H} = \hbar\omega a^\dagger a + \frac{\mathrm{i}\hbar}{2}(\varepsilon a^{\dagger 2} - \varepsilon^* a^2) + a\Gamma^\dagger + a^\dagger\Gamma \,, \tag{7.53}$$

where $\varepsilon = \varepsilon_{\mathrm{p}}\chi$ and ε_{p} is the amplitude of the pump, and χ is proportional to the nonlinear susceptibility of the medium. Γ is the reservoir operator representing cavity losses. We consider here the case of a single ended cavity with loss rate γ_1.

The Heisenberg equations of motion for $\boldsymbol{a}(t)$ are linear and given by (7.20) where

$$A = \begin{pmatrix} \frac{\gamma_1}{2} & -\varepsilon \\ -\varepsilon^* & \frac{\gamma_1}{2} \end{pmatrix} \,. \tag{7.54}$$

We can obtain an expression for the Fourier components of the output field from (7.26)

$$\begin{aligned} a_{\mathrm{OUT}}(\omega) = & \frac{1}{\left[\left(\frac{\gamma_1}{2} - \mathrm{i}\omega\right)^2 - |\varepsilon|^2\right]} \left\{ \left[\left(\frac{\gamma_1}{2}\right)^2 + \omega^2 + |\varepsilon|^2\right] \right. \\ & \left. \times a_{\mathrm{IN}}(\omega) + \varepsilon\gamma_1 a_{\mathrm{IN}}^\dagger(-\omega) \right\} \,. \end{aligned} \tag{7.55}$$

Defining the quadrature phase operators by

$$2a_{\mathrm{OUT}} = \mathrm{e}^{\mathrm{i}\theta/2}(X_1^{\mathrm{OUT}} + \mathrm{i}X_2^{\mathrm{OUT}}) \,, \tag{7.56}$$

where θ is the phase of the pump, we find the following correlations:

$$\langle : X_1^{\mathrm{OUT}}(\omega), X_1^{\mathrm{OUT}}(\omega') : \rangle = \frac{2\gamma_1 |\varepsilon|}{\left(\frac{\gamma_1}{2} - |\varepsilon|\right)^2 + \omega^2} \delta(\omega + \omega') , \tag{7.57}$$

$$\langle : X_2^{\mathrm{OUT}}(\omega), X_2^{\mathrm{OUT}}(\omega') : \rangle = \frac{-2\gamma_1 |\varepsilon|}{\left(\frac{\gamma_1}{2} + |\varepsilon|\right)^2 + \omega^2} \delta(\omega + \omega') , \tag{7.58}$$

where the input field a_{IN} has been taken to be in the vacuum.

The δ function in (7.57 and 7.58) may be removed by integrating over ω' to give the normally ordered spectrum: $S^{\mathrm{OUT}}(\omega)$:. The final result for the squeezing spectra of the quadrature is

$$S_1^{\mathrm{OUT}}(\omega) = 1 + : S_1^{\mathrm{OUT}}(\omega) := 1 + \frac{2\gamma_1 |\varepsilon|}{\left(\frac{\gamma_1}{2} - |\varepsilon|\right)^2 + \omega^2} , \tag{7.59}$$

$$S_2^{\mathrm{OUT}}(\omega) = 1 + : S_2^{\mathrm{OUT}}(\omega) := 1 - \frac{2\gamma_1 |\varepsilon|}{\left(\frac{\gamma_1}{2} + |\varepsilon|\right)^2 + \omega^2} , \tag{7.60}$$

These spectra are defined in a frame of frequency Ω so that $\omega = 0$ is on cavity resonance.

The maximum squeezing occurs at the threshold for parametric oscillation $|\varepsilon| = \gamma_1/2$ where

$$S_1^{\mathrm{OUT}}(\omega) = 1 + \left(\frac{\gamma_1}{\omega}\right)^2 , \tag{7.61}$$

$$S_2^{\mathrm{OUT}}(\omega) = 1 - \frac{\gamma_1^2}{\gamma_1^2 + \omega^2} , \tag{7.62}$$

Thus the squeezing occurs in the X_2 quadrature which is $\pi/2$ out of phase with the pump. The light generated in parametric oscillation is therefore said to be phase squeezed.

In Fig. 7.3 we plot $S_2^{\mathrm{OUT}}(\omega)$ at threshold. We see that at $\omega = 0$, that is the cavity resonance, the fluctuations in the X_2 quadrature tend to zero. The fluctuations in the X_1 quadrature on the other hand diverge at $\omega = 0$. This is characteristic of critical fluctuations which diverge at a critical point. In this case however the critical

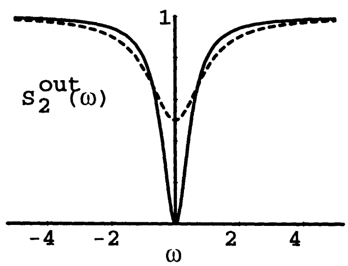

Fig. 7.3 A plot of the spectrum of the squeezed quadrature for a cavity containing a parametric amplifier with a classical pump. *Solid*: single-sided cavity with $\gamma_1 = \gamma_2$, *dashed*: double-sided cavity

flucuations are phase dependent. As the fluctuations in one phase are reduced to zero the fluctuations in the other phase necessarily diverge. This characteristic of good squeezing near critical points is found in other phase dependent nonlinear optical systems [2]. This behaviour is in contrast to the threshold for laser oscillation where the critical fluctuations are random in phase.

7.7 Squeezing in the Total Field

The squeezing in the total field may be found by integrating (7.62) over ω. At threshold we find

$$S_2^{\mathrm{TOT}} = \int \left(1 - \frac{\gamma_1^2}{\gamma_1^2 + \omega^2}\right) \mathrm{d}\omega = \frac{\gamma_1}{2} \, . \tag{7.63}$$

The squeezing in the total field is given by the equal time correlation functions

$$\begin{aligned} \langle a, a \rangle_{\mathrm{OUT}} &= \gamma_1 \langle a, a \rangle , \\ \langle a, a^\dagger \rangle_{\mathrm{OUT}} &= \gamma_1 \langle a, a^\dagger \rangle \, . \end{aligned} \tag{7.64}$$

Hence, the squeezing in the internal field is

$$V(X_2) = \frac{1}{2} \, . \tag{7.65}$$

Thus the internal field mode is 50% squeezed, in agreement with the calculations of *Milburn* and *Walls* [3]. This can be surpassed in the individual frequency components of the output field which have 100% squeezing for $\omega = 0$. It is the squeezing in the individual frequency components of the output field which may be measured by a spectrum analyser following a homodyne detection scheme.

7.8 Fokker–Planck Equation

We shall now give an alternative method for evaluating the squeezing spectrum. This converts the operator master equation to a c-number Fokker–Planck equation. This is a useful technique when the operator equations are nonlinear. Standard linearization techniques for the fluctuations may be made in the Fokker–Planck equation. We shall consider applications of this technique to nonlinear systems in Chap. 8.

We shall first demonstrate how time and normally-ordered moments may be calculated directly using the P representation. We consider the following time- and normally-ordered moment

$$T\langle : X_1(t)X_1(0) :\rangle = \mathrm{e}^{-2\mathrm{i}\theta}\langle a(t)a(0)\rangle + \mathrm{e}^{2\mathrm{i}\theta}\langle a^\dagger(0)a^\dagger(t)\rangle \\ + \langle a^\dagger(t)a(0)\rangle + \langle a^\dagger(0)a(t)\rangle \, . \tag{7.66}$$

The two-time correlation functions may be evaluated using the P representation which determines normally-ordered moments. Thus equal time moments of the c-number variables give the required normally-ordered operator moments. The two time moments imply precisely the time ordering of the internal operators that are required to compute the output moments. This can be seen by noting that the evolution of the system will in general mix $a^\dagger$ and a. Hence $a(t+\tau)$ contains both $a(t)$ and $a^\dagger(t), \;\tau > 0$. In a normally-ordered two time product $a(t+\tau)$ must therefore stand to the left of $a(t)$, similarly $a^\dagger(t+\tau)$ must stand to the right of $a^\dagger(t)$. Thus

$$\langle \alpha(t+\tau)\alpha(t)\rangle_P = \langle a(t+\tau)a(t)\rangle \, , \tag{7.67}$$

$$\langle \alpha^*(t+\tau)\alpha^*(t)\rangle_P = \langle a^\dagger(t)a^\dagger(t+\tau)\rangle \, , \tag{7.68}$$

where the left-hand side of these equations represent averages of c-number variables over the P representation. The normally-ordered output correlation matrix defined by

$$: C^{\mathrm{OUT}}(\tau) := \begin{pmatrix} \langle a_{\mathrm{OUT}}(t+\tau), a_{\mathrm{OUT}}(t)\rangle & \langle a^\dagger_{\mathrm{OUT}}(t), a_{\mathrm{OUT}}(t+\tau)\rangle \\ \langle a^\dagger_{\mathrm{OUT}}(t+\tau), a_{\mathrm{OUT}}(t)\rangle & \langle a^\dagger_{\mathrm{OUT}}(t+\tau), a^\dagger_{\mathrm{OUT}}(t)\rangle \end{pmatrix} \tag{7.69}$$

is given by

$$: C^{\mathrm{OUT}}(\tau) : = \gamma \begin{pmatrix} \langle \alpha(t+\tau), \alpha(t)\rangle & \langle \alpha(t+\tau), \alpha^*(t)\rangle \\ \langle \alpha^*(t+\tau), \alpha(t)\rangle & \langle \alpha^*(t+\tau), \alpha^*(t)\rangle \end{pmatrix} \\ \equiv \gamma C_P(\tau) \, . \tag{7.70}$$

The two time correlation functions for the output field may be calculated directly from the correlation functions of the stochastic variables describing the internal field using the P representation.

For nonlinear optical processes the Fokker–Planck equation for the P function may have nonlinear drift terms and nonconstant diffusion. In such circumstances we first linearise the equation about the deterministic steady states, to obtain a linear Fokker–Planck equation of the form

$$\frac{\partial P}{\partial t}(\alpha) = \left(\frac{\partial}{\partial \alpha_i} A_i \alpha_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} D_{ij} \right) P(\alpha) \, , \tag{7.71}$$

where A is the drift matrix, and D is the diffusion matrix. The linearised description is expected to give the correct descriptions away from instabilities in the deterministic equations of motion. For fields exhibiting quantum behaviour, such as squeezing, D is non-positive definite and a Fokker–Planck equation is not defined for the Glauber–Sudarshan P function. In these cases a Fokker–Planck equation is defined for the positive P representation, where α^* is replaced by $\alpha^\dagger$ an independent complex variable as described in Chap. 6.

The spectral matrix $S(\omega)$ is defined as the Fourier transform of $C_p(\tau)$. In a linearised analysis it is given by

$$S(\omega) = \gamma(\boldsymbol{A} + \mathrm{i}\omega\boldsymbol{I})^{-1} D(\boldsymbol{A}^T - \mathrm{i}\omega\boldsymbol{I})^{-1} . \tag{7.72}$$

The squeezing spectrum for each quadrature phase is then given by

$$: S_1^{\mathrm{OUT}}(\omega) := \gamma[\mathrm{e}^{-2\mathrm{i}\theta} S_{11}(\omega) + \mathrm{e}^{2\mathrm{i}\theta} S_{22}(\omega) + S_{12}(\omega) + S_{21}(\omega)] \tag{7.73}$$

$$: S_2^{\mathrm{OUT}}(\omega) := \gamma[-\mathrm{e}^{-2\mathrm{i}\theta} S_{11}(\omega) - \mathrm{e}^{2\mathrm{i}\theta} S_{22}(\omega) + S_{12}(\omega) + S_{21}(\omega)] \tag{7.74}$$

These spectra are defined in a frame of frequency Ω (the cavity-resonance frequency) so that $\omega = 0$ corresponds to the cavity resonance.

It should be noted that in the above derivation there is only one input field and one output field, that is, there is only one source of cavity loss. Thus the above results only apply to a single-ended cavity; one in which losses accrue only at one mirror.

If there are other significant losses from the cavity the γ appearing in (7.60 and 7.61) is not the total loss but only the loss from the mirror through which the output field of interest is transmitted.

The above procedure enables one to calculate the squeezing in the output field from an optical cavity, provided the internal field may be described by the linear Fokker–Planck equation (7.71).

Alternatively the squeezing spectrum for the parametric oscillator may be calculated using the Fokker–Planck equation. The Fokker–Planck equation for the distribution $P(\alpha)$ for the system described by the Hamiltonian (7.53) may be derived using the techniques of Chap. 6.

$$\begin{aligned} \frac{\partial P(\alpha)}{\partial t} = -\Bigg\{ & \left(\varepsilon^* \frac{\partial}{\partial \alpha^*}\alpha + \varepsilon \frac{\partial}{\partial \alpha}\alpha^* \right) + \frac{\gamma_1}{2}\left(\frac{\partial}{\partial \alpha^*}\alpha^* + \frac{\partial}{\partial \alpha}\alpha \right) \\ & + \frac{1}{2}\left[\varepsilon^* \frac{\partial^2}{\partial \alpha^{*2}} + \varepsilon \frac{\partial^2}{\partial \alpha^2} \right] \Bigg\} P(\alpha) \end{aligned} \tag{7.75}$$

The drift and diffusion matrices are

$$A = \begin{pmatrix} \frac{\gamma_1}{2} & -\varepsilon \\ -\varepsilon^* & \frac{\gamma_1}{2} \end{pmatrix}, \quad D = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^* \end{pmatrix} . \tag{7.76}$$

Direct application of (7.72–7.74) yields the squeezing spectra given by (7.59 and 7.60).

Exercises

7.1 Calculate the squeezing spectrum for a degenerate parametric oscillator with losses γ_1 and γ_2 at the end mirrors.

7.2 Calculate the squeezing spectrum for a non-degenerate parametric oscillator. [Hint: Use the quadratures for a two mode system described in (5.49)].

References

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Further Reading

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